

# Limit laws of the coefficients of polynomials with only unit roots

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## Abstract

We consider sequences of random variables whose probability generating functions are polynomials all of whose roots lie on the unit circle. The distribution of such random variables has only been sporadically studied in the literature. We show that the random variables are asymptotically normally distributed if and only if the fourth normalized (by the standard deviation) central moment tends to 3, in contrast to the common scenario for polynomials with only real roots for which a central limit theorem holds if and only if the variance goes unbounded. We also derive a representation theorem for all possible limit laws and apply our results to many concrete examples in the literature, ranging from combinatorial structures to numerical analysis, and from probability to analysis of algorithms.

## 1 Introduction

The close connection between the location of the zeros of a function (or a polynomial) and the distribution of its coefficients has long been the subject of extensive study; typical examples include the order of an entire function and its zeros in Analysis, and the limit distribution of the coefficients of polynomials when all roots are real in Combinatorics, Probability and Statistical Physics. We address in this paper the situation when the roots of the sequence of probability generating functions all lie on the unit circle. While one may convert the situation with only unimodular zeros to that with only real zeros by a suitable change of variables, such *root-unitary polynomials* turn out to have many fascinating properties due mainly to the boundedness of all zeros. In particular, we show that the fourth normalized central moments

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are (asymptotically) always bounded between 1 and 3, the limit distribution being Bernoulli if they tend to 1 and Gaussian if they tend to 3.

Although this class of polynomials does not have a standard name, we will refer to them as, following [Kedlaya \(2008\)](#) and for convention, root-unitary polynomials. Other related terms include *self-inversive* (zeros symmetric in the unit circle), *reciprocal* or *self-reciprocal* ( $P(z) = z^n P(z^{-1})$ ), *uni-modular* (all coefficients of modulus one), *palindromic* ( $a_j = a_{n-j}$ ), etc., when  $P(z) = \sum_{0 \leq j \leq n} a_j z^j$  is a polynomial of degree  $n$ .

Unit roots of polynomials play a very special and important role in many scientific and engineering disciplines, notably in statistics and signal processing where the unit root test decides if a time series variable is non-stationary. On the other hand, many nonparametric statistics are closely connected to partitions of integers, which lead to generating functions whose roots all lie on the unit circle. We will discuss many examples in Sections 4 and 5. Another famous example is the Lee-Yang partition function for Ising model, which has stimulated a widespread interest in the statistical-physical literature since the 1950's.

While there is a large literature on polynomials with only real zeros, the distribution of the coefficients of root-unitary polynomials has only been sporadically studied; more references will be given below. It is well known that for polynomials with nonnegative coefficients whose roots are all real, one can decompose the polynomials into products of linear factors, implying that the associated random variables are expressible as sums of independent Bernoulli random variables. Thus one obtains a *Gaussian limit law for the coefficients if and only if the variance tends to infinity*; see the survey paper [Pitman \(1997\)](#) for more information and for finer estimates. A representative example is the Stirling numbers of the second kind for which [Harper \(1967\)](#) showed that the generating polynomials have only real roots<sup>1</sup>; he also established the asymptotic normality of these numbers by proving that the variance tends to infinity. For more examples and information on polynomials with only real roots, see [Brenti \(1994\)](#), [Pitman \(1997\)](#) and the references therein. See also [Haigh \(1971\)](#), [Hayman \(1956\)](#), [Rényi \(1972\)](#) for different extensions.

Our first main result states that if we restrict the range where the roots of the polynomials  $P_n(z)$  can occur to the unit circle  $|z| = 1$ , then the asymptotic normality of  $X_n$  defined by the coefficients is determined by the limiting behavior of its fourth normalized central moment. *Throughout this paper, write  $X_n^* := (X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$ .*

**Theorem 1.1.** *Let  $\{X_n\}$  be a sequence of discrete random variables whose probability generating functions  $\mathbb{E}(z^{X_n})$  are polynomials of degree  $n$  with all roots  $\rho_j$  lying on the unit circle  $|\rho_j| = 1$ .*

– (Bounds for the fourth normalized central moment)

$$1 \leq \mathbb{E}(X_n^*)^4 < 3 \quad (n \geq 1). \quad (1)$$

– (Asymptotic normality) *The sequence of random variables  $\{X_n^*\}$  converges in distribution (and with all moments) to the standard normal law  $\mathcal{N}(0, 1)$  if and only if*

$$\mathbb{E}(X_n^*)^4 \rightarrow 3 \quad (n \rightarrow \infty). \quad (2)$$

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<sup>1</sup>The fact that the Stirling polynomials (of the second kind) have only real roots had been known long before Harper; see for example [d'Ocagne \(1887\)](#); in addition, [Bell \(1938\)](#) wrote (without providing reference) that all results of d'Ocagne's paper were already obtained thirty years before him by a number of English authors.

- (Asymptotic Bernoulli distribution) The sequence  $\{X_n^*\}$  converges to Bernoulli random variable assuming the two values  $-1$  and  $1$  with equal probabilities if and only if

$$\mathbb{E}(X_n^*)^4 \rightarrow 1 \quad (n \rightarrow \infty). \quad (3)$$

This theorem implies that Gaussian and Bernoulli distributions are in a certain sense extremal limit laws for the distribution of  $X_n$ , maximizing and minimizing asymptotically the value of the fourth moment  $\mathbb{E}(X_n^*)^4$ , respectively, with other limit laws lying in between.

A standard example where Gaussian limit law arises is the number of inversions in random permutations (or Kendall's  $\tau$ -statistic)

$$P_{\binom{n}{2}}(z) = \prod_{1 \leq k \leq n} \frac{1 + z + \dots + z^{k-1}}{k}.$$

A straightforward calculation shows that the fourth normalized central moment has the form

$$3 - \frac{9(6n^2 + 15n + 16)}{25n(n-1)(n+1)},$$

which implies the asymptotic normality by Theorem 1.1; see Feller (1945), Section 4 for more details and examples.

On the other hand, a Bernoulli limit law results from the simple example

$$P_{2n}(z) = \frac{1 + z^{2n}}{2}.$$

It is then natural to ask to which limit laws other than normal and Bernoulli can the sequence of random variables  $X_n^*$  converge. The simplest such example is the uniform distribution

$$P_{2n}(z) = \frac{1 + z + z^2 + \dots + z^{2n}}{2n + 1};$$

or, more generally, the finite sum of uniform distributions

$$P_{n_1 + \dots + n_k}(z) = \prod_{1 \leq j \leq k} \frac{1 + z + \dots + z^{2n_j}}{2n_j + 1}.$$

Observe that the moment generating functions of the above three distributions have the following representations.

- Normal distribution:  $e^{s^2/2}$ ;
- Bernoulli distribution (assuming  $\pm 1$  with equal probability):

$$\frac{e^s + e^{-s}}{2} = \cos(is) = \prod_{k \geq 1} \left( 1 + \frac{4s^2}{(2k-1)^2 \pi^2} \right);$$

- Uniform distribution (with zero mean and unit variance):

$$\frac{1}{2\sqrt{3}} \int_{\sqrt{3}}^{\sqrt{3}} e^{xs} dx = \frac{\sin(\sqrt{3} is)}{\sqrt{3} is} = \prod_{k \geq 1} \left( 1 + \frac{3s^2}{k^2 \pi^2} \right).$$

Here we used the well-known expansions (see [Titchmarsh \(1975\)](#))

$$\cos s = \prod_{k \geq 1} \left( 1 - \frac{4s^2}{(2k-1)^2\pi^2} \right), \quad \frac{\sin s}{s} = \prod_{k \geq 1} \left( 1 - \frac{s^2}{k^2\pi^2} \right).$$

We show that these are indeed special cases of a more general representation theorem for the limit laws.

**Theorem 1.2.** *Let  $\{X_n\}$  be a sequence of random variables whose probability generating functions are polynomials with only roots of modulus one. If the sequence  $\{X_n^*\}$  converges to some limit distribution  $X$ , then the moment generating function of  $X$  is finite and has the infinite-product representation*

$$\mathbb{E}(e^{Xs}) = e^{qs^2/2} \prod_{k \geq 1} \left( 1 + \frac{q_k}{2} s^2 \right), \quad (4)$$

where  $q$  and the sequence  $\{q_k\}$  are all non-negative numbers such that

$$q + \sum_{k \geq 1} q_k = 1.$$

The above examples show that  $q_k = \frac{8}{\pi^2(2k-1)^2}$  for Bernoulli distribution and  $q_k = \frac{6}{\pi^2 k^2}$  for the uniform distribution. More examples will be discussed below.

It remains open to characterize infinite-product representations of the form (4) that are themselves the moment generating functions of limit laws of root-unitary polynomials. On the other hand, many sufficient criteria for root-unitarity have been proposed in the literature; see, for example, the book [Milovanović et al. \(1994\)](#) and the recent papers [Lalin and Smyth \(2012\)](#); [Suzuki \(2012\)](#) for more information and references.

This paper is organized as follows. We first prove Theorem 1.1 in the next section when  $n$  is even, and then modify the proof to cover polynomials of odd degrees. Theorem 1.2 is then proved in Section 3. We then apply the results to many concrete examples from the literature: Section 4 for normal limit laws and Section 5 for non-normal laws. A class of polynomials with non-normal limit law is included in Appendix since the root-unitarity property has not yet been proved.

## 2 Moments and the two extremal limit distributions

For convenience, we begin by considering (general) polynomials of even degree with all their roots lying on the unit circle

$$P_{2n}(z) = \sum_{0 \leq k \leq 2n} p_k z^k,$$

where  $p_k \geq 0$ . To avoid triviality, we assume that not all  $p_k$ 's are zero. Observe that if  $|\rho| = 1$  and  $P(\rho) = 0$ , then  $P(\bar{\rho}) = 0$ . If  $\rho = 1$ , then its multiplicity must be even since all other roots can be grouped in pairs and are symmetric with respect to the real line. Thus our polynomials can be factored as

$$P_{2n}(z) = \prod_{1 \leq j \leq n} (z - \rho_j)(z - \bar{\rho}_j),$$

where  $|\rho_j| = 1$  for  $j = 1, \dots, n$ . This factorization implies that root-unitary polynomials are always self-inversive.

**Lemma 2.1.** *The coefficients of a root-unitary polynomial of even degree  $2n$  are symmetric with respect to  $n$ , that is*

$$p_{n-k} = p_{n+k} \quad (0 \leq k \leq n).$$

*Proof.* By replacing  $z$  by  $1/z$ , we get

$$\begin{aligned} \sum_{0 \leq k \leq 2n} p_{2n-k} z^k &= z^{2n} P_{2n}(1/z) = \prod_{1 \leq j \leq n} (1 - z\rho_j)(1 - z\bar{\rho}_j) \\ &= \prod_{1 \leq j \leq n} (z - \rho_j)(z - \bar{\rho}_j) = P_{2n}(z) = \sum_{0 \leq k \leq 2n} p_k z^k. \end{aligned}$$

Taking the coefficients of  $z^k$  on both sides, we obtain  $p_{2n-k} = p_k$  for  $0 \leq k \leq 2n$ , which proves the lemma. ■

## 2.1 Random variables, moments and cumulants

Since the coefficients of  $P_{2n}(z)$  are nonnegative, we can define a random variable  $X_{2n}$  by

$$\mathbb{E}(z^{X_{2n}}) = \frac{P_{2n}(z)}{P_{2n}(1)}.$$

For convenience, we write  $\rho_j = e^{i\phi_j}$  since  $|\rho_j| = 1$ . Then

$$(z - \rho_j)(z - \bar{\rho}_j) = 1 - (\rho_j + \bar{\rho}_j)z + z^2 = 1 - 2z \cos \phi_j + z^2.$$

It follows that

$$\mathbb{E}(z^{X_{2n}}) = \prod_{1 \leq j \leq n} \frac{1 - 2z \cos \phi_j + z^2}{2(1 - \cos \phi_j)}. \quad (5)$$

Note that  $\phi_j \neq 0$  for  $1 \leq j \leq n$  since  $P_{2n}(1) > 0$ .

It turns out that the mean values of such random variables are identically  $n$ .

**Lemma 2.2.** *For  $n \geq 1$*

$$\mathbb{E}(X_{2n}) = n. \quad (6)$$

*Proof.* By (5), take derivative with respect to  $z$  and then substitute  $z = 1$ . ■

The relation (6) indeed holds more generally for self-inversive polynomials; see, for example, [Sheil-Small \(2002\)](#).

**Corollary 2.3.** *All odd central moments of  $X_{2n}$  are zero*

$$\mathbb{E}(X_{2n} - n)^{2m+1} = 0 \quad (m = 0, 1, \dots).$$

*Proof.* This follows from the symmetry of the coefficients  $p_k$ . ■

For even moments, we look at the cumulants, which are defined as

$$\mathbb{E}(e^{(X_{2n}-n)s}) = \exp \left( \sum_{m \geq 1} \frac{\kappa_m(n)}{m!} s^m \right),$$

where  $\kappa_{2m+1}(n) = 0$ .

**Lemma 2.4.** *The  $2m$ -th cumulant  $\kappa_{2m}(n)$  of  $X_{2n}$  is given by*

$$\kappa_{2m}(n) = (2m)! \sum_{1 \leq k \leq m} \frac{(-1)^{k-1}}{k 2^k} h_{m,k} S_{n,k} \quad (m \geq 1), \quad (7)$$

where  $2^{2k} \sinh^{2k}(s/2) = \sum_{m \geq k} h_{m,k} s^{2m}$ , with  $h_{k,k} = 1$ , and

$$S_{n,k} := \sum_{1 \leq j \leq n} \frac{1}{(1 - \cos \phi_j)^k}.$$

*Proof.* By (5), we have

$$\begin{aligned} \log \frac{1 - 2e^s \cos \phi + e^{2s}}{2(1 - \cos \phi)} &= \log \left( e^s + \frac{(e^s - 1)^2}{2(1 - \cos \phi)} \right) \\ &= s + \log \left( 1 + 2 \frac{\sinh^2(s/2)}{1 - \cos \phi} \right). \end{aligned}$$

Thus

$$\log \mathbb{E}(e^{(X_{2n}-n)s}) = \sum_{1 \leq j \leq n} \log \left( 1 + 2 \frac{\sinh^2(s/2)}{(1 - \cos \phi_j)} \right),$$

which implies (7) by a direct expansion. ■

## 2.2 Variance and fourth central moment

In particular, we obtain, from (7),

$$\sigma_n^2 := \mathbb{V}(X_{2n}) = \kappa_2(n) = \sum_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j}. \quad (8)$$

**Lemma 2.5.** *The variance satisfies the inequalities*

$$\frac{n}{2} \leq \sigma_n^2 \leq n^2. \quad (9)$$

*Proof.* The lower bound follows from (8) and the inequality  $1 - \cos \phi_j \leq 2$ . The upper bound is also straightforward

$$\mathbb{V}(X_{2n}) = \frac{1}{P_{2n}(1)} \sum_{0 \leq k \leq 2n} p_k (k - n)^2 \leq n^2,$$

which shows that the distance of the unit zeroes of  $P_{2n}(z)$  to the point 1 is always larger than  $c/n$ , where  $c > 0$  is an absolute constant. ■

On the other hand, by the elementary inequalities

$$\frac{2}{\pi^2} t^2 \leq 1 - \cos t \leq \frac{t^2}{2} \quad (t \in [-\pi, \pi]), \quad (10)$$

we have

$$2 \leq \frac{\sigma_n^2}{\sum_{1 \leq j \leq n} \phi_j^{-2}} \leq \frac{\pi^2}{2}.$$

We now turn to the fourth central moment. Define

$$\omega_n := \frac{1}{\sigma_n^4} \sum_{1 \leq j \leq n} \frac{1}{(1 - \cos \phi_j)^2}. \quad (11)$$

**Lemma 2.6.** (i) For  $n \geq 1$ ,

$$1 \leq \mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^4 \leq 3 - \frac{1}{2\sigma_n^2} < 3. \quad (12)$$

(ii)

$$\mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^4 \rightarrow 3 \quad \text{iff} \quad \omega_n \rightarrow 0. \quad (13)$$

*Proof.* By definition and by (7),

$$\mathbb{E} \left( \frac{X_{2n} - n}{\sigma} \right)^4 = 3 + \frac{\kappa_4(n)}{\sigma_n^4} = 3 + \sigma_n^{-2} - 3\omega_n. \quad (14)$$

Now

$$\sigma_n^{-2} - 3\omega_n = -\frac{1}{\sigma_n^4} \sum_{1 \leq j \leq n} \frac{2 + \cos \phi_j}{(1 - \cos \phi_j)^2} \leq -\frac{1}{2\sigma_n^4} \sum_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j} = -\frac{1}{2}\sigma_n^{-2} < 0,$$

proving the upper bound of (12). On the other hand, since  $1/\sigma_n \leq \sqrt{2/n}$  (by (9)), we see that (13) also holds. It remains to prove the lower bound of (12), which results directly from the Cauchy-Schwarz inequality

$$1 = \mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^2 \leq \left( \mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^4 \right)^{1/2}.$$

■

By (10), we can replace the condition  $\omega_n \rightarrow 0$  by

$$\sum_{1 \leq j \leq n} \phi_j^{-4} = o \left( \left( \sum_{1 \leq j \leq n} \phi_j^{-2} \right)^2 \right).$$

On the other hand,  $\mathbb{E}((X_n - n)/\sigma_n)^4 \rightarrow 3$  is equivalent to  $\kappa_4(n)/\kappa_2^2(n) \rightarrow 0$ ; the latter condition is in many cases easier to manipulate.

Note that (12) proves (1) when  $n$  is even.

### 2.3 Estimates for the moment generating functions

**Lemma 2.7.** Assume  $\omega_n \rightarrow 0$ . For all  $s \in \mathbb{C}$  such that  $|s| \leq \min\{\sigma_n, \omega_n^{-1/4}\}/4$ , we have

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) = \exp\left(\frac{s^2}{2} + O\left(\frac{|s|^3}{\sigma_n} + \omega_n |s|^4\right)\right). \quad (15)$$

*Proof.* By (5),

$$\log \mathbb{E}(e^{X_{2n}s/\sigma_n}) = \sum_{1 \leq j \leq n} \log\left(1 + (e^{s/\sigma_n} - 1) + \frac{(e^{s/\sigma_n} - 1)^2}{2(1 - \cos \phi_j)}\right)$$

Note that, by (8),

$$\sigma_n^2 \geq \max_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j}.$$

From the definition (11) of  $\omega_n$ , we also have

$$\frac{1}{\sigma_n^4} \max_{1 \leq j \leq n} \frac{1}{(1 - \cos \phi_j)^2} \leq \omega_n,$$

which means that

$$\max_{1 \leq j \leq n} \frac{1}{1 - \cos \phi_j} \leq \sigma_n^2 \sqrt{\omega_n}. \quad (16)$$

Thus

$$\left| e^{s/\sigma_n} - 1 + \frac{(e^{s/\sigma_n} - 1)^2}{2(1 - \cos \phi_j)} \right| \leq e^{2|s|/\sigma_n} \left( \frac{|s|}{\sigma_n} + |s|^2 \sqrt{\omega_n} \right)$$

Since  $\omega_n \rightarrow 0$  by assumption, the right-hand side is less than, say  $2/3$ , for large enough  $n$  when  $|s|$  remains bounded. Thus we can use the Taylor expansion of  $\log(1 + w)$  and obtain

$$\begin{aligned} \log\left(1 + (e^{s/\sigma_n} - 1) + \frac{(e^{s/\sigma_n} - 1)^2}{2(1 - \cos \phi_j)}\right) \\ = \frac{s}{\sigma_n} + \frac{s^2}{2\sigma_n^2(1 - \cos \phi_j)} + O\left(\frac{|s|^3}{\sigma_n^3(1 - \cos \phi_j)} + \frac{|s|^4}{\sigma_n^4(1 - \cos \phi_j)^2} + \frac{|s|^6}{\sigma_n^6(1 - \cos \phi_j)^3}\right). \end{aligned}$$

By (16)

$$\frac{|s|^6}{\sigma_n^6(1 - \cos \phi_j)^3} \leq \frac{|s|^6 \sqrt{\omega_n}}{\sigma_n^4(1 - \cos \phi_j)^2}.$$

It follows, after summing over all  $j$ , that

$$\log \mathbb{E}(e^{X_{2n}s/\sigma_n}) = \frac{s^2}{2} + \frac{ns}{\sigma_n} + O\left(\frac{|s|^3}{\sigma_n} + (|s|^4 + |s|^6 \sqrt{\omega_n})\omega_n\right).$$

Now if  $|s| \leq \min\{\sigma_n, \omega_n^{-1/4}\}/4$ , then  $\omega_n^{3/2}|s|^6 \leq \omega_n |s|^4/16$ , and this proves (15). ■

**Lemma 2.8.** For  $s \in \mathbb{R}$ , the inequality

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) \leq \exp\left(\frac{3}{2}s^2 e^{2s/\sigma_n}\right) \quad (17)$$

holds.



*Proof.* By (5) and the elementary inequality  $1 + y \leq e^y$  for real  $y$ , we obtain

$$\begin{aligned}\mathbb{E}(z^{X_{2n}}) &= \prod_{1 \leq j \leq n} \left( 1 + (z - 1) + \frac{(z - 1)^2}{2(1 - \cos \phi_j)} \right) \\ &\leq \prod_{1 \leq j \leq n} \exp \left( z - 1 + \frac{(z - 1)^2}{2(1 - \cos \phi_j)} \right) \\ &= e^{n(z-1) + \sigma_n^2(z-1)^2/2}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) &\leq \exp \left( n \left( e^{s/\sigma_n} - 1 - \frac{s}{\sigma_n} \right) + \frac{1}{2} \sigma_n^2 (e^{s/\sigma_n} - 1)^2 \right) \\ &\leq \exp \left( \frac{n}{2\sigma_n^2} s^2 e^{s/\sigma_n} + \frac{s^2}{2} e^{2s/\sigma_n} \right),\end{aligned}$$

and (17) follows from the inequality  $n/\sigma_n^2 \leq 2$ . ■

## 2.4 Normal limit law

We now prove the second part of Theorem 1.1 in the case of polynomials of even degree, namely,  $\{(X_n - n)/\sigma_n\}$  converges in distribution and with all moments to the standard normal distribution if and only if

$$\mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^4 \rightarrow 3.$$

*Proof.* Consider first the sufficiency part. By (13),  $\omega_n \rightarrow 0$ , and we can apply the estimate (15), implying the convergence in distribution of  $(X_{2n} - n)/\sigma_n$  to  $\mathcal{N}(0, 1)$ .

On the other hand, by Lemma 2.8,

$$\mathbb{E}(e^{(X_{2n}-n)s/\sigma_n}) = \sum_{m \geq 0} \left( \frac{X_{2n} - n}{\sigma_n} \right)^{2m} \frac{s^{2m}}{(2m)!} \leq e^{\frac{3}{2}s^2 e^{2s/\sigma_n}}.$$

Taking  $s = 1$ , we conclude that all normalized central moments of  $X_{2n}$  are bounded above by

$$\mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^{2m} \leq (2m)! e^{\frac{3}{2} e^{2/\sigma_n}}.$$

Thus we also have convergence of all moments.

For the necessity, we see that if  $\{(X_{2n} - n)/\sigma_n\}$  converges in distribution to  $\mathcal{N}(0, 1)$ , then the fact that the moments of  $(X_{2n} - n)/\sigma_n$  are all bounded implies that all the normalized central moments of  $X_{2n}$  converge to the moments of the standard normal distribution; in particular, the fourth normalized central moments converge to 3. ■

## 2.5 Bernoulli limit law

We now examine the case when the fourth moment converges to the smallest possible value, that is

$$\mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^4 \rightarrow 1. \quad (18)$$

Note that

$$\mathbb{V} \left( \frac{X_{2n} - n}{\sigma_n} \right)^2 = \mathbb{E} \left( \left( \frac{X_{2n} - n}{\sigma_n} \right)^2 - 1 \right)^2 = \mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^4 - 1.$$

If (18) holds, then by Chebyshev's inequality, we see that

$$\mathbb{P} \left( \frac{X_{2n} - n}{\sigma_n} \in (-1 - \varepsilon, -1 + \varepsilon) \cup (1 - \varepsilon, 1 + \varepsilon) \right) \rightarrow 1,$$

for any  $\varepsilon > 0$ . By symmetry of the random variable  $X_{2n} - n$

$$\mathbb{P} \left( \frac{X_{2n} - n}{\sigma_n} \in (-1 - \varepsilon, -1 + \varepsilon) \right) = \mathbb{P} \left( \frac{X_{2n} - n}{\sigma_n} \in (1 - \varepsilon, 1 + \varepsilon) \right).$$

We conclude that the distributions of  $(X_{2n} - n)/\sigma_n$  converge to a Bernoulli distribution that assumes the two values 1 and  $-1$  with equal probability.

## 2.6 Polynomials of odd degree

To complete the proof of Theorem 1.1, we need to address the situation of odd-degree polynomials.

Assume  $Q_{2n-1}(z)$  is a root-unitary polynomial of degree  $2n - 1$  with non-negative coefficients. If we multiply it by the factor  $1 + z$ , then the resulting polynomial

$$P_{2n}(z) = (1 + z)Q_{2n-1}(z)$$

remains root-unitary with non-negative coefficients. This means that the moment generating functions of the corresponding random variables  $\mathbb{E}(e^{Y_{2n-1}s}) := Q_{2n-1}(e^s)/Q_{2n-1}(1)$  and  $\mathbb{E}(e^{X_{2n}s}) := P_{2n}(e^s)/P_{2n}(1)$  are connected by the identity

$$\mathbb{E}(e^{X_{2n}s}) = \frac{1 + e^s}{2} \mathbb{E}(e^{Y_{2n-1}s}).$$

This leads to the relation

$$X_{2n} \stackrel{d}{=} Y_{2n-1} + B, \quad (19)$$

where  $B$  is independent of  $Y_{2n-1}$  and takes the values 0 and 1 with equal probability. Thus

$$\begin{aligned} \mathbb{E}(Y_{2n-1}) &= \mathbb{E}(X_{2n}) - \frac{1}{2} = n - \frac{1}{2}, \\ \mathbb{V}(Y_{2n-1}) &= \mathbb{V}(X_{2n}) - \frac{1}{4} = \sigma_n^2 - \frac{1}{4}, \end{aligned} \quad (20)$$

and

$$\mathbb{E} (Y_{2n-1} - \mathbb{E}(Y_{2n-1}))^4 = \mathbb{E}(X_{2n} - n)^4 - \frac{3}{2}\sigma_n^2 + \frac{5}{16}. \quad (21)$$

Thus we obtain

$$\mathbb{E} \left( \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} \right)^4 \leq \frac{\sigma_n^4}{(\sigma_n^2 - \frac{1}{4})^2} \mathbb{E} \left( \frac{X_{2n} - n}{\sqrt{\mathbb{V}(X_{2n})}} \right)^4 - \frac{\frac{3}{2}\sigma_n^2 - \frac{5}{16}}{(\sigma_n^2 - \frac{1}{4})^2},$$

which, by (12), is bounded above by

$$\frac{\sigma_n^4}{(\sigma_n^2 - \frac{1}{4})^2} \left( 3 - \frac{1}{\sigma_n^2} \right) - \frac{\frac{3}{2}\sigma_n^2 - \frac{5}{16}}{(\sigma_n^2 - \frac{1}{4})^2} = 3 - \frac{1}{\sigma_n^2} - \frac{6\sigma_n^2 - 1}{\sigma_n^2(4\sigma_n^2 - 1)^2} \leq 3 - \sigma_n^{-2} < 3.$$

Thus the fourth normalized central moment is bounded above by 3; the lower bound follows from the same Cauchy-Schwarz inequality used in the even-degree cases.

On the other hand, since (again by (19))

$$\frac{X_{2n} - \mathbb{E}(X_{2n})}{\sqrt{\mathbb{V}(X_{2n})}} \stackrel{d}{=} \frac{\sqrt{\mathbb{V}(Y_{2n-1})}}{\sqrt{\mathbb{V}(X_{2n})}} \cdot \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} + \frac{B - \frac{1}{2}}{\sqrt{\mathbb{V}(X_{2n})}},$$

we have, by (20),

$$\frac{X_{2n} - \mathbb{E}(X_{2n})}{\sqrt{\mathbb{V}(X_{2n})}} \stackrel{d}{=} \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} \left( 1 + O \left( \frac{1}{\sqrt{n}} \right) \right) + O(\sigma_n^{-1}). \quad (22)$$

The last identity implies that both sides converge to the same limit law.

Assume that the fourth central moment of  $Y_{2n-1}$  satisfies

$$\mathbb{E} \left( \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} \right)^4 \rightarrow 3. \quad (23)$$

Then, by (21), we obtain

$$\mathbb{E} \left( \frac{X_{2n} - \mathbb{E}(X_{2n})}{\sqrt{\mathbb{V}(X_{2n})}} \right)^4 = \left( \frac{\mathbb{V}(Y_{2n-1})}{\mathbb{V}(X_{2n})} \right)^2 \mathbb{E} \left( \frac{Y_{2n-1} - \mathbb{E}(Y_{2n-1})}{\sqrt{\mathbb{V}(Y_{2n-1})}} \right)^4 + O(\sigma_n^{-1}).$$

Thus the left-hand side also tends to 3 and, consequently,  $X_{2n}$  is asymptotically normally distributed. The asymptotic distribution of  $X_{2n}$  then implies, by (22), that of  $Y_{2n-1}$ .

The proof for the Bernoulli case is similar and is omitted.

### 3 The infinite-product representation for general limit laws

We first prove Theorem 1.2 in this section, and then mention some of its consequences.

### 3.1 Proof of Theorem 1.2

It suffices to consider only the sequence of polynomials of even degree. The symmetry of distribution of the limit law  $X$  follows from the symmetry of coefficients of polynomials  $P_{2n}(z)$ . The inequality (17) for the moment generating function of  $(X_{2n} - n)/\sigma_n$  implies that the moment generating function of the limit distribution  $X$  is also finite, and thus  $X$  is uniquely determined by its moments. This means that the sequence  $\{(X_{2n} - n)/\sigma_n\}$  converges in distribution to  $X$  as  $n \rightarrow \infty$  if and only if

$$\mathbb{E} \left( \frac{X_{2n} - n}{\sigma_n} \right)^m \rightarrow \mathbb{E}(X^m) \quad (m \geq 0),$$

as  $n \rightarrow \infty$ . Thus the cumulant  $\bar{\kappa}_m(n)$  of  $(X_{2n} - n)/\sigma_n$  of order  $m$  also converges to the cumulant of  $X$  of order  $m$  for  $m \geq 1$ . Note that  $\bar{\kappa}_{2m+1}(n) = 0$  for  $m \geq 0$  and (see (7))

$$\bar{\kappa}_{2m}(n) = \sigma_n^{-2m} \kappa_{2m}(n) = \frac{(2m)!}{\sigma_n^{2m}} \sum_{1 \leq k \leq m} \frac{(-1)^{k-1}}{k 2^k} h_{m,k} S_{n,k}.$$

Since  $S_{n,k} \leq \sigma_n^{2k}$ , we deduce that

$$\frac{\bar{\kappa}_{2m}(n)}{(2m)!} = \frac{(-1)^{m-1}}{m 2^m} \cdot \frac{S_{n,m}}{\sigma_n^{2m}} + O(\sigma_n^{-2}),$$

for any fixed  $m$ . Now  $\sigma_n \rightarrow \infty$ , we conclude that

$$\frac{\kappa_{2m}}{(2m)!} = \lim_{n \rightarrow \infty} \frac{\bar{\kappa}_{2m}(n)}{(2m)!} = \frac{(-1)^{m-1}}{m 2^m} \lim_{n \rightarrow \infty} \frac{S_{n,m}}{\sigma_n^{2m}}. \quad (24)$$

We now introduce the distribution function

$$F_n(x) := \sum_{\frac{1}{\sigma_n^2(1-\cos \phi_j)} \leq x} \frac{1}{\sigma_n^2(1-\cos \phi_j)},$$

with support in the unit interval. Then

$$\frac{S_{n,N}}{\sigma_n^{2N}} = \int_0^1 x^{N-1} dF_n(x).$$

The fact that the left-hand side of the above expression has a limit (24) implies that the corresponding sequence of distribution functions  $F_n(x)$  also converges weakly to some limit distribution function  $F(x)$ . Therefore

$$\lim_{n \rightarrow \infty} \frac{S_{n,N}}{\sigma_n^{2N}} = \int_0^1 x^{N-1} dF(x),$$

which implies that the cumulants of the limit distribution  $X$  can be expressed as

$$\frac{\bar{\kappa}_{2m}}{(2m)!} = \lim_{n \rightarrow \infty} \frac{\bar{\kappa}_{2m}(n)}{(2m)!} = \frac{(-1)^{m-1}}{m 2^m} \lim_{n \rightarrow \infty} \frac{S_{n,m}}{\sigma_n^{2m}} = \frac{(-1)^{m-1}}{m 2^m} \int_0^1 x^{m-1} dF(x).$$

It follows that

$$\begin{aligned}
\mathbb{E}(e^{Xs}) &= \exp \left( \sum_{m \geq 1} \frac{\kappa_{2m}}{(2m)!} s^{2m} \right) \\
&= \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1} s^{2m}}{m 2^m} \int_0^1 x^{m-1} dF(x) \right) \\
&= \exp \left( \int_0^1 \frac{\log(1 + xs^2/2)}{x} dF(x) \right).
\end{aligned} \tag{25}$$

Note that the distribution function  $F_n(x)$  has no more than  $\lfloor 1/\varepsilon \rfloor$  points of discontinuity in the interval  $[\varepsilon, 1]$  if  $\varepsilon > 0$ . Thus the weak limit  $F(x)$  of the sequence of  $F_n(x)$  has the same property:  $F(x)$  has no more than  $\lfloor 1/\varepsilon \rfloor$  points of discontinuity  $q_k$  in the interval  $[\varepsilon, 1]$ , where  $q_k$  is the limit of certain points of discontinuity of function  $F_n(x)$ . This means that  $F(x)$  is a distribution function of the form

$$F(x) = \begin{cases} q + \sum_{q_k < x} q_k, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

where  $q_k > 0$  with  $\sum_{k \geq 1} q_k = 1 - q$ . Here  $q$  equals the jump of the function  $F(x)$  at zero. Thus

$$\int_0^1 \frac{\log(1 + xs^2/2)}{x} dF(x) = \frac{q}{2} s^2 + \sum_{k \geq 1} \log \left( 1 + \frac{q_k}{2} s^2 \right).$$

Substituting this expression into (25), we obtain (4). This completes the proof of Theorem 1.2.

### 3.2 An alternative proof of Theorem 1.2

A less elementary proof of Theorem 1.2 relies on the Hadamard factorization theorem (see (Titchmarsh, 1975, Ch. 8); see also Newman (1974) for a similar context). Indeed, assume that  $(X_{2n} - n)/\sigma_n$  converges in distribution to some limit law  $X$ , then the inequality (17) implies that

$$|\mathbb{E}(e^{Xs})| \leq e^{3|s|^2/2} \quad (s \in \mathbb{C}).$$

In other words, it is an entire function of order 2. Hadamard's factorization theorem then implies that such a function can be represented as an infinite product

$$\mathbb{E}(e^{Xs}) = e^{As^2 + Bs} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho},$$

where  $\rho$  ranges over all zeros of the function of the left-hand side. On the other hand, the fact that all zeroes of the functions  $\mathbb{E}(e^{(X_{2n} - n)s/\sigma_n})$  are symmetrically located on the imaginary line implies the same property for  $\mathbb{E}(e^{Xs})$ . This yields

$$\mathbb{E}(e^{Xs}) = e^{As^2 + Bs} \prod_{k \geq 1} \left( 1 + \frac{s^2}{t_k^2} \right),$$

for some real sequence  $t_k > 0$ . Now  $\mathbb{E}(X) = 0$  implies that  $B = 0$ . Also  $\mathbb{E}(X^2) = 1$  leads to

$$A + \sum_{k \geq 1} t_k^{-2} = 1.$$

Denoting by  $q = 2A$  and  $q_k = 2/t_k^2$ , we obtain the representation (4).

### 3.3 Implications of the infinite-product factorization

By (4),

$$\kappa_{2m} = \frac{(-1)^{m-1}}{m2^m} \sum_{j \geq 1} q_j^m \quad (m \geq 2).$$

This yields the sign-alternating property for the sequence  $\{\kappa_{2m}\}$ .

**Corollary 3.1.** *If  $X$  is not the normal law, then all even cumulants are non-zero and have alternating signs*

$$(-1)^{m-1} \kappa_{2m} > 0 \quad (m \geq 1).$$

**Corollary 3.2.**

$$1 \leq \mathbb{E}(X^4) \leq 3.$$

*Proof.* By (4),

$$\mathbb{E}(X^4) = 3 \left( 1 - \sum_{j \geq 1} q_j^2 \right), \quad (26)$$

which implies the upper bound; the lower bound follows directly from Cauchy-Schwarz inequality  $1 = \mathbb{E}(X^2) \leq \sqrt{\mathbb{E}(X^4)}$ . ■

**Corollary 3.3.** *The standard normal distribution is the only distribution for which the fourth moment reaches the maximum value 3 in the class of distributions that are the limits of random variables whose probability generating functions are root-unitary polynomials; similarly, the Bernoulli distribution assuming  $\pm 1$  with probability  $1/2$  each is the only distribution whose fourth moment reaches the minimum value 1 in the same class of distributions.*

*Proof.* Note that the standard normal law corresponds to the choices  $q = 1$  and  $q_j \equiv 0$ , the first part of the corollary follows then from (26).

For the lower bound, assume that  $Y$  is a symmetric distribution such that  $\mathbb{E}(Y) = 0$  and  $\mathbb{E}(Y^2) = \mathbb{E}(Y^4) = 1$ . Then

$$\mathbb{V}(Y^2) = \mathbb{E}(Y^2 - 1)^2 = \mathbb{E}(Y^4 - 2Y^2 + 1) = 0.$$

This means that  $Y$  can only assume two values  $\mathbb{P}(Y \in \{-1, 1\}) = 1$ . The symmetry of  $Y$  now implies that  $Y$  assumes the values 1 and  $-1$  with equal probabilities. ■

*Remark 3.4.* The uniqueness of the standard normal and Bernoulli laws also implies that a sequence of random variables  $\{X_n\}$  converges to normal or Bernoulli if and only if its fourth normalized central moment converges to 3 or to 1, respectively. This provides an alternative proof of the last two statements of Theorem 1.1.

## 4 Applications. I. Normal limit law

We consider in this section applications of our results in the situations when the limit law is normal.

## 4.1 A simple framework

Our starting point is the polynomials of the form

$$P_n(z) = \frac{(1 - z^{b_1})(1 - z^{b_2}) \cdots (1 - z^{b_N})}{(1 - z^{a_1})(1 - z^{a_2}) \cdots (1 - z^{a_N})}, \quad (27)$$

where  $a_j, b_j$  are non-negative integers that may depend themselves on  $N$  and

$$n := \sum_{1 \leq j \leq N} (b_j - a_j).$$

We assume that  $P_n(z)$  has only nonnegative coefficients. Such a simple form arises in a large number of diverse contexts, some of which will be examined below. In particular, it was studied in the recent paper [Chen et al. \(2008\)](#).

We now consider a sequence of random variables  $X_n$  defined by

$$\mathbb{E}(z^{X_n}) = \frac{P_n(z)}{P_n(1)}.$$

We have

$$\frac{P_n(e^s)}{P_n(1)} = \exp\left(\sum_{m \geq 1} \frac{\kappa_{N,m}}{m!} s^m\right),$$

where

$$\kappa_{N,m} = \frac{(-1)^m}{m} B_m \sum_{1 \leq j \leq N} (b_j^m - a_j^m) \quad (m \geq 1),$$

the  $B_m$ 's being the Bernoulli numbers. Note that  $B_{2m+1} = 0$  for  $m \geq 1$ .

An application of Theorem 1.1 yields the following result.

**Theorem 4.1.** *The sequence of the random variables  $(X_n - \mathbb{E}(X_n))/\sqrt{\mathbb{V}(X_n)}$  converges to the standard normal distribution if and only if the following cumulant condition holds*

$$\lim_{N \rightarrow \infty} \frac{\kappa_{N,4}}{\kappa_{N,2}^2} = \frac{144}{120} \lim_{N \rightarrow \infty} \frac{\sum_{1 \leq j \leq N} (b_j^4 - a_j^4)}{\left(\sum_{1 \leq j \leq N} (b_j^2 - a_j^2)\right)^2} = 0. \quad (28)$$

The cumulant condition largely simplifies the sufficient condition given by [Chen et al. \(2008\)](#), where they require the convergence of all cumulants

$$\frac{\kappa_{N,2m}}{\kappa_{N,2}^m} \rightarrow 0 \quad (m \geq 2),$$

following the proof used by [Sachkov \(1997\)](#). See also [Janson \(1988\)](#) for a related framework.

## 4.2 Applications of Theorem 4.1

Theorem 4.1 can be applied to a large number of examples. Many other examples related to Poincaré polynomials, rank statistics, and integer partitions can be found in the literature; see, for example, [Akyıldız \(2004\)](#); [Andrews \(1976\)](#); [van de Wiel et al. \(1999\)](#) and the references therein.

**Inversions in permutations** The generating polynomial for the number of inversions in a permutation of  $n$  elements (or Kendall's  $\tau$  statistic) is given by

$$\prod_{1 \leq j \leq n} \frac{1 - z^j}{1 - z}.$$

In this case, the cumulant condition (28) has the form

$$\frac{\sum_{1 \leq j \leq n} (j^4 - 1)}{\left(\sum_{1 \leq j \leq n} (j^2 - 1)\right)^2} = O(n^{-1}).$$

Thus the number of inversions in random permutations is asymptotically normally distributed; see [Feller \(1945\)](#), [Sachkov \(1997\)](#); see also [Cronholm and Revusky \(1965\)](#); [Louchard and Prodinger \(2003\)](#); [Margolius \(2001\)](#).

**Number of inversions in Stirling permutations** In this case, we have the polynomial (see [Park \(1994\)](#))

$$\prod_{1 \leq j \leq n} \frac{1 - z^{r+(j-1)r^2}}{1 - z^r} \quad (r \geq 1),$$

and the cumulant condition (28) is of order

$$\frac{\kappa_{n,4}}{\kappa_{n,2}^2} = \frac{\sum_{0 \leq j < n} ((r + jr^2)^4 - 1)}{\left(\sum_{0 \leq j < n} ((r + jr^2)^2 - 1)\right)^2} = O(n^{-1}).$$

Consequently, the number of inversions in random Stirling permutations is asymptotically normally distributed.

**Gaussian polynomials** The generating function for the number  $p(n, m, j)$  of partitions of integer  $j$  into at most  $m$  parts, each  $\leq n$ , is given by (see e.g. [Andrews \(1976\)](#))

$$\sum_{0 \leq j \leq nm} p(n, m, j) z^j = \prod_{1 \leq j \leq n} \frac{1 - z^{j+m}}{1 - z^j}.$$

Then the cumulant condition has the form

$$\frac{\sum_{1 \leq j \leq n} ((m + j)^4 - j^4)}{\left(\sum_{1 \leq j \leq n} ((m + j)^2 - j^2)\right)^2} = O\left(\frac{1}{m} + \frac{1}{n}\right).$$

This means that the coefficients of Gaussian polynomials are normally distributed if both  $n, m \rightarrow \infty$ ; see [Mann and Whitney \(1947\)](#); [Takács \(1986\)](#). More examples can be found in [Andrews \(1976\)](#).



**Mahonian statistics** In this case the polynomials are equal to the general  $q$ -multinomial coefficients (see [Canfield et al. \(2011\)](#) and [Canfield et al. \(2012\)](#))

$$P_n(z) = \frac{\prod_{1 \leq j \leq a_1 + \dots + a_m} (1 - z^j)}{\prod_{1 \leq j \leq m} \prod_{1 \leq i \leq a_j} (1 - z^i)},$$

where  $n = \sum_{2 \leq k \leq m} a_k \sum_{1 \leq j < k} a_j$ . By symmetry, we can assume that  $a_1 \geq \dots \geq a_m$ . Then the cumulant condition (28) becomes

$$\frac{\sum_{1 \leq j \leq a_1 + \dots + a_m} i^4 - \sum_{1 \leq j \leq m} \sum_{1 \leq i \leq a_j} i^4}{\left( \sum_{1 \leq j \leq a_1 + \dots + a_m} i^2 - \sum_{1 \leq j \leq m} \sum_{1 \leq i \leq a_j} i^2 \right)^2} = \frac{f_4(a_1 + \dots + a_m) - \sum_{1 \leq j \leq m} f_4(a_j)}{\left( f_2(a_1 + \dots + a_m) - \sum_{1 \leq j \leq m} f_2(a_j) \right)^2},$$

where  $f_2(x) = (2x^3 + 3x^2 + x)/6$  and  $f_4(x) = (6x^5 + 15x^4 + 10x^3 - x)/30$ . By induction,  $(a_1 + \dots + a_m)^k - a_1^k - \dots - a_m^k$  is nonnegative and is nondecreasing in  $k \geq 1$ . Thus the right-hand side is bounded above by

$$\begin{aligned} & \frac{9 \cdot 31}{30} \frac{(a_1 + \dots + a_m)^5 - a_1^5 - \dots - a_m^5}{\left( (a_1 + \dots + a_m)^3 - a_1^3 - \dots - a_m^3 \right)^2} \\ &= O \left( \frac{a_1 + \dots + a_m}{\sum_{1 \leq i < j \leq m} a_i a_j} \right) = O \left( \frac{a_1 + \dots + a_m}{a_1(a_2 + a_3 + \dots + a_m)} \right) \\ &= O \left( \frac{1}{a_2 + a_3 + \dots + a_m} + \frac{1}{a_1} \right), \end{aligned}$$

where we use the estimates

$$\begin{aligned} (a_1 + \dots + a_m)^3 - a_1^3 - \dots - a_m^3 &\asymp (a_1 + \dots + a_m) \sum_{1 \leq i < j \leq m} a_i a_j, \\ (a_1 + \dots + a_m)^5 - a_1^5 - \dots - a_m^5 &\asymp (a_1 + \dots + a_m)^3 \sum_{1 \leq i < j \leq m} a_i a_j. \end{aligned}$$

Thus we arrive at the same conditions as those in [Canfield et al. \(2011\)](#)

$$a_1 \rightarrow \infty \quad \text{and} \quad a_2 + a_3 + \dots + a_m \rightarrow \infty,$$

for the asymptotic normality of the coefficients of  $P_n(z)$  when  $a_1 \geq a_2 \geq \dots \geq a_m$ .

**Generalized  $q$ -Catalan numbers** The generating function has the form

$$\prod_{2 \leq j \leq n} \frac{1 - z^{(m-1)n+j}}{1 - z^j},$$

and the cumulant condition (28) also holds

$$\frac{\sum_{2 \leq j \leq n} (((m-1)n+j)^4 - j^4)}{\left( \sum_{2 \leq j \leq n} (((m-1)n+j)^2 - j^2) \right)^2} \leq \frac{\sum_{2 \leq j \leq n} (2mn)^4}{\left( \sum_{2 \leq j \leq n} (m-1)^2 n^2 \right)^2} = O(n^{-1}),$$

which means that the generalized  $q$ -Catalan numbers are asymptotically normally distributed, uniformly for all  $m \geq 2$ . This result was previously proved by [Chen et al. \(2008\)](#).

**Sums of uniform discrete distributions** Let  $X_n$  be the sum of  $N$  independent, integer-valued random variables

$$X_n := J_1 + J_2 + \cdots + J_N,$$

where  $J_k$  is a uniform distribution on the set  $\{0, 1, 2, \dots, d_k - 1\}$  with  $d_k \geq 2$ , and  $n = \sum_{1 \leq j \leq N} (d_j - 1)$ . Then the corresponding probability generating function  $\mathbb{E}(z^{X_n})$  is equal, up to a normalizing constant, to

$$P_n(z) = \prod_{1 \leq j \leq N} \frac{1 - z^{d_j}}{1 - z},$$

which means that  $X_n$  is asymptotically normal if and only if

$$\frac{\sum_{1 \leq j \leq N} (d_j^4 - 1)}{\left(\sum_{1 \leq j \leq N} (d_j^2 - 1)\right)^2} \rightarrow 0.$$

Since by our assumption  $d_j \geq 2$ , we have  $d_j - 1 \asymp d_j$  and thus we can simplify our necessary and sufficient condition for asymptotic normality as

$$\frac{d_1^4 + d_2^4 + \cdots + d_N^4}{(d_1^2 + d_2^2 + \cdots + d_N^2)^2} \rightarrow 0 \quad (N \rightarrow \infty). \quad (29)$$

Note that  $d_j$  here can depend on  $N$ . The continuous version of this problem with  $J_k$  being uniformly distributed on the intervals  $[0, d_j]$  was considered in [Olds \(1952\)](#). The corresponding necessary and sufficient condition obtained in this paper was

$$\frac{\max_{1 \leq j \leq N} d_j}{\sqrt{d_1^2 + d_2^2 + \cdots + d_N^2}} \rightarrow 0$$

which is equivalent to condition (29).

**Number of inversions in bimodal permutations** A permutation  $\sigma = (s_1, s_2, \dots, s_n)$  of  $n$  numbers  $1, 2, 3, \dots, n$  is said to be of a shape  $(i, k - j, j, l)$  if the first  $i$  numbers in the permutation are decreasing  $s_1 > s_2 > \cdots > s_i$ , the next  $k - j$  numbers are increasing  $s_{i+1} > s_2 > \cdots > s_{i+k-j}$ , then followed by  $j$  increasing and  $l$  decreasing numbers. Assume that  $\sigma$  is chosen with equal probability among all permutations of shape  $(i, k - j, j, l)$ . Then its number of inversions becomes a random variable  $I_n = I_n(i, k - j, j, l)$ . The probability generating function of  $I_n$  is, up to some constant, of the form (see [Böhm and Katzenbeisser \(2005\)](#))

$$P_n(i, k, l, j; z) = z^{\binom{i}{2} + \binom{j}{2}} \left( \prod_{1 \leq \nu \leq i} \frac{1 - z^{k+\nu}}{1 - z^\nu} \right) \left( \prod_{1 \leq \nu \leq l} \frac{1 - z^{k+i+\nu}}{1 - z^\nu} \right) \left( \prod_{1 \leq \nu \leq j} \frac{1 - z^{k-j+\nu}}{1 - z^\nu} \right).$$

The random variables  $I_n$  are asymptotically normally distributed if

$$\frac{\sum_{\nu=1}^i ((k+\nu)^4 - \nu^4) + \sum_{\nu=1}^l ((k+i+\nu)^4 - \nu^4) + \sum_{\nu=1}^j ((k-j+\nu)^4 - \nu^4)}{\left(\sum_{\nu=1}^i ((k+\nu)^2 - \nu^2) + \sum_{\nu=1}^l ((k+i+\nu)^2 - \nu^2) + \sum_{\nu=1}^j ((k-j+\nu)^2 - \nu^2)\right)^2} \rightarrow 0,$$

which is equivalent to

$$\frac{ik(k+i)^3 + l(k+i)(k+i+l)^3 + j(k^4 - j^4)}{(ik(k+i) + l(k+i)(k+i+l) + j(k^2 - j^2))^2} \rightarrow 0.$$

If we assume that the parameters  $i, j, k, l$  are proportionate to some parameter  $t$ , that is  $i = \lfloor \alpha t \rfloor, j = \lfloor \beta t \rfloor, k = \lfloor \gamma t \rfloor, l = \lfloor \delta t \rfloor$ , where  $\alpha, \beta, \gamma, \delta > 0$  and  $\alpha + \gamma + \delta = 1$ , then the above condition is satisfied and as a consequence  $I_n$  is asymptotically normally distributed as  $t \rightarrow \infty$ . This fact has been proved in [Böhm and Katzenbeisser \(2005\)](#) by the method of moments.

**Rank statistics** Many test statistics based on ranks lead to explicit generating functions that are of the form (27), and thus the corresponding limit distribution can be dealt with by the tools we established. In particular, we have the following correspondence between test statistics and combinatorial structures; see [van de Wiel et al. \(1999\)](#) for more information.

Kendall's $\tau$	Inversions in permutations
Mann-Whitney test	Gaussian polynomials
Jonckheere-Terpstra test	Mahonian statistics

On the other hand, the Wilcoxon signed rank test (see [Wilcoxon \(1947\)](#)) leads to the probability generating function of the form

$$\prod_{1 \leq j \leq n} \frac{1 + z^j}{2},$$

which admits a straightforward generalization to (see [van de Wiel et al. \(1999\)](#) for details)

$$\prod_{1 \leq j \leq n} \frac{1 + z^{a_j}}{2},$$

where the  $a_j$ 's can be any real numbers. When they are all nonnegative integers, we see, by (28), that the associated random variables are asymptotically normally distributed if and only if

$$\frac{a_1^4 + \cdots + a_n^4}{(a_1^2 + \cdots + a_n^2)^2} \rightarrow 0,$$

as  $n \rightarrow \infty$ . In particular, this applies to Wilcoxon's test ( $a_j = j$ ) and to Policello and Hettmansperger's test ( $a_j = \min\{2j, n + 1\}$ ; [Policello and Hettmansperger \(1976\)](#)).

### 4.3 Turán-Fejér polynomials

The class of polynomials we consider here (see (30) below) is of interest for several reasons. First, they lead to asymptotically normally distributed random variables but do not have the finite-product form (27). Second, they provide natural examples with non-normal limit laws when the second parameter varies. Finally, they have a concrete interpretation in terms of the partitioning cost of some variants of quicksort.

[Fejér \(1937\)](#) studied the Cesàro summation of the geometric series defined by

$$F_{n,k}(z) := \sum_{0 \leq j \leq n} F_{j,k-1}(z) \quad (k \geq 1),$$

with

$$F_{n,0}(z) := \sum_{0 \leq j \leq n} z^j,$$

and [Turán \(1949\)](#) proved that all  $F_{n,k}^{(k)}(z)$  are root-unitary for  $0 \leq k \leq n$ . We characterize all possible limit laws for the random variables defined via the coefficients of  $F_{n,k}^{(k)}(z)$  for  $0 \leq k \leq n$ .

By the relation

$$F_{n,k}(z) = [w^n] \frac{1}{(1-w)^{k+1}(1-zw)},$$

where  $[w^n]f(w)$  denotes the coefficient of  $w^n$  in the Taylor expansion of  $f(w)$ , we have

$$\begin{aligned} F_{n,k}^{(k)}(z) &= [w^n] \frac{1}{(1-w)^{k+1}} \cdot \frac{k!w^k}{(1-zw)^{k+1}} \\ &= k! \sum_{0 \leq j \leq n-k} \binom{j+k}{k} \binom{n-j}{k} z^j. \end{aligned}$$

Normalizing this polynomial, we obtain

$$P_{n,k}(z) := \sum_{0 \leq j \leq n-k} \frac{\binom{j+k}{k} \binom{n-j}{k}}{\binom{n+k+1}{2k+1}} z^j, \quad (30)$$

which gives rise to a sequence of probability generating functions of random variables, say  $Z_{n,k}$ . Note that

$$z^k P_{n-k-1,k}(z) = \sum_{k \leq j \leq n-k-1} \frac{\binom{j}{k} \binom{n-1-j}{k}}{\binom{n}{2k+1}} z^j,$$

which arises in the analysis of quicksort using the median of  $2k+1$  elements; see [Sedgewick \(1980\)](#); [Chern et al. \(2002\)](#) or Appendix.

**Lemma 4.2.** *For  $m \geq 0$*

$$\mathbb{E}(Z_{n,k}^m) = \sum_{0 \leq \ell \leq m} S(m, \ell) \ell! \frac{\binom{k+\ell}{k} \binom{n+k+1}{2k+\ell+1}}{\binom{n+k+1}{2k+1}}, \quad (31)$$

where  $S(m, \ell)$  denotes the Stirling numbers of the second kind. In particular,

$$\mathbb{E}(Z_{n,k}) = \frac{n-k}{2} \quad \text{and} \quad \mathbb{V}(Z_{n,k}) = \frac{(n-k)(n+k+2)}{4(2k+3)}. \quad (32)$$

*Proof.* By (30), the relation

$$j^m = \sum_{0 \leq \ell \leq m} S(m, \ell) j \cdots (j - \ell + 1),$$

and the combinatorial identity

$$\sum_{0 \leq j \leq n-k} \binom{j+k}{k} \binom{n-j}{k} \binom{j}{\ell} = \binom{k+\ell}{k} \binom{n+k+1}{2k+\ell+1},$$

(easily proved by convolution), we deduce (31).  $\blacksquare$

**Theorem 4.3.** *The random variables  $Z_{n,k}$  are asymptotically normally distributed if and only if both  $k$  and  $n - k$  tend to infinity. If  $0 \leq k = O(1)$ , then the limit law is a Beta distribution*

$$\frac{Z_{n,k}}{n} \xrightarrow{d} \text{Beta}(k, k). \quad (33)$$

If  $1 \leq \ell := n - k = O(1)$ , then the limit law is a binomial distribution

$$Z_{n,k} \xrightarrow{d} \text{Binom}(\ell; \frac{1}{2}).$$

*Proof.* By (32), the variance tends to infinity if and only if  $n - k \rightarrow \infty$  ( $0 \leq k \leq n$ ). Also we obtain, by (31),

$$\frac{\mathbb{E} \left( Z_{n,k} - \frac{n-k}{2} \right)^4}{\mathbb{V}(Z_{n,k})^2} - 3 = -\frac{2(3n^2 + 6n + k^2 + 4k + 6)}{(n-k)(n+k+2)(2k+5)} = O\left(\frac{n}{k(n-k)}\right).$$

The asymptotic normality then follows. We can indeed obtain a local limit theorem by straightforward calculations from (30).

When  $k = O(1)$ , we have, by (30),

$$\frac{\mathbb{E}(Z_{n,k}^m)}{n^m} \rightarrow \frac{(k+m)!(2k+1)!}{k!(2k+m+1)!} \quad (m \geq 0),$$

implying that the moment generating function of the limit law satisfies

$$\mathbb{E}(e^{Z_k s}) = \frac{(2k+1)!}{k!} \sum_{m \geq 0} \frac{(k+m)!}{m!(2k+m+1)!} s^m = \frac{(2k+1)!}{k!k!} \int_0^1 x^k (1-x)^k e^{xs} dx,$$

a Beta distribution. Note that we can express the moment generating function in terms of Bessel functions as

$$\begin{aligned} \mathbb{E}(e^{(Z_k - 1/2)s}) &= \left(\frac{is}{4}\right)^{-k-1/2} \Gamma(k + \frac{3}{2}) J_{k+\frac{1}{2}}(is/2) \\ &= \prod_{j \geq 1} \left(1 + \frac{s^2}{4\zeta_{k+1/2,j}^2}\right), \end{aligned} \quad (34)$$

where  $J_\alpha$  denotes the Bessel function and the  $\zeta_{\alpha,j}$ 's denote the positive zeros of  $J_\alpha(z)$  arranged in increasing order. By considering  $2(Z_k - 1/2)\sqrt{2k+3}$ , we obtain (4) with  $q_j = 2(2k+3)/\zeta_{k+1/2,j}^2$ .

On the other hand, when  $\ell := n - k = O(1)$ , we have, by (31),

$$P_{n,k}(z) \rightarrow \left(\frac{1+z}{2}\right)^\ell,$$

a binomial distribution. Note that we have the factorization

$$\mathbb{E}(e^{(X - \ell/2)s/\sqrt{\ell/4}}) = \prod_{j \geq 1} \left(1 + \frac{4s^2}{(2j-1)^2 \pi^2 \ell}\right)^\ell.$$

■

## 5 Applications II. Non-normal limit laws

In addition to the extremal cases of the Turán-Fejér polynomials, we consider in this section more root-unitary polynomials whose coefficients have a limit distribution that is not Gaussian. A class of polynomials exhibiting a similar non-Gaussian behavior is included in Appendix because the proof that they are root-unitary is still missing.

### 5.1 Reimer's polynomials

In the course of investigating the remainder theory of finite difference, [Reimer \(1969\)](#) proved, as a side result, that the polynomials

$$R_{n,m}(y) := \sum_{0 \leq j \leq n} \binom{n}{j} y^j \int_j^{j+1} |t(t-1) \cdots (t-n-1)|^m dt.$$

have only unit roots. We consider the distribution of the coefficients of  $R_{n,m}(y)$ .

For simplicity, we consider only  $m = 1$  and write  $R_n = R_{n,1}$ . Define the random variables  $X_n$  by

$$\mathbb{E}(y^{X_n}) := \frac{R_n(y)}{R_n(1)}.$$

Let

$$A_k := [z^k] \frac{z}{\log(1-z)} \quad (k \geq 0).$$

These numbers are (up to sign) known under the name of Cauchy numbers; see ([Comtet, 1974](#), pp. 293–294). See also the recent paper [Kowalenko \(2010\)](#) for a detailed study of these numbers.

**Lemma 5.1.** *For  $n \geq 1$*

$$\mathbb{E}(y^{X_n}) = 12 \sum_{0 \leq j \leq n} \binom{n}{j} y^{n-j} (1-y)^j A_{j+2}. \quad (35)$$

*Proof.* We have

$$\begin{aligned} R_n(y) &= \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^{n+1+j} y^j \int_j^{j+1} t(t-1) \cdots (t-n-1) dt \\ &= (n+2)! (-1)^{n+1} \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j y^j \int_j^{j+1} \binom{t}{n+2} dt \\ &= (n+2)! (-1)^{n+1} [z^{n+2}] \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j y^j \int_j^{j+1} (1+z)^t dt \\ &= (n+2)! (-1)^{n+1} [z^{n+1}] \sum_{0 \leq j \leq n} \binom{n}{j} (-1)^j y^j \frac{(1+z)^j}{\log(1+z)} \\ &= (n+2)! [z^{n+1}] \frac{(1 - (1-z)y)^n}{\log(1-z)}. \end{aligned} \quad (36)$$

In particular

$$R_n(1) = (n+2)! [z] \frac{1}{\log(1-z)} = \frac{(n+2)!}{12},$$

and (35) follows. ■

Note that  $A_0 = -1$  and

$$A_k = - \sum_{0 \leq j < k} \frac{A_j}{k+1-j} \quad (k \geq 1).$$

All  $A_k$ 's are positive except  $A_0$ .

**Lemma 5.2.** *The moments of  $X_n$  satisfy*

$$\mathbb{E}(X_n^m) = \sum_{0 \leq k \leq m} \tilde{A}_k S(m, k) n(n-1) \cdots (n-k+1) \quad (m \geq 0), \quad (37)$$

where

$$\tilde{A}_k := 12 \sum_{0 \leq \ell \leq k} \binom{k}{\ell} (-1)^\ell A_{\ell+2}. \quad (38)$$

In particular,

$$\mathbb{E}(X_n) = \frac{n}{2}, \quad \mathbb{V}(X_n) = \frac{n}{60}(4n+11).$$

*Proof.* By taking  $m$ -th derivative with respect to  $y$  and then substituting  $y = 1$  in (36), we obtain

$$\begin{aligned} \mathbb{E}(X_n(X_n-1) \cdots (X_n-m+1)) &= 12[z^{n+1}] \frac{\partial^m}{\partial y^m} \frac{(1-(1-z)y)^n}{\log(1-z)} \Big|_{y=1} \\ &= 12[z^{m+1}] \frac{(z-1)^m}{\log(1-z)} n(n-1) \cdots (n-m+1), \end{aligned}$$

which yields (37) since

$$\mathbb{E}(X_n^m) = \sum_{0 \leq k \leq m} S(m, k) \mathbb{E}(X_n(X_n-1) \cdots (X_n-k+1)).$$

■

**Theorem 5.3.** *The sequence of random variables  $\{X_n/n\}$  converges in distribution to  $X$  whose  $m$ -th moment equals  $\tilde{A}_m$  (defined in (38)).*

*Proof.* By (37),  $\mathbb{E}(X_n^m) \sim \tilde{A}_m n^m$ . Since  $A_k = O(1/k)$ , we see that  $\tilde{A}_m = O(2^m)$ , implying that such a moment sequence determines uniquely a distribution. ■

The limit law has the moment generating function

$$\begin{aligned}
\mathbb{E}(e^{Xs}) &= 12 \sum_{m \geq 0} \frac{s^m}{m!} \sum_{0 \leq j \leq m} \binom{m}{j} (-1)^j A_{j+2} \\
&= 12e^s \sum_{j \geq 0} \frac{A_{j+2}}{j!} (-s)^j \\
&= -\frac{12}{2\pi i s} \int_{\mathcal{H}} e^{t+s} \left( \frac{1}{\log(1 + \frac{s}{t})} - \frac{t}{s} - \frac{1}{2} \right) dt,
\end{aligned}$$

where the integration  $\int_{\mathcal{H}}$  is taken along some Hankel contour; see (Flajolet and Sedgewick, 2009, p. 745).

When  $m \geq 2$ , we can apply the same arguments but the technicalities are more involved.

## 5.2 Chung-Feller's arcsine law

The classical Chung-Feller theorem states that the number of positive terms  $W_n$  of the sums  $S_n = X_1 + \dots + X_n$ , where  $X_i$  takes  $\pm 1$  with probability  $1/2$  each, has the probability

$$\mathbb{P}(W_n = k) = \binom{2k}{k} \binom{2n-2k}{n-k} 4^{-n} \quad (k = 0, \dots, n).$$

The limit distribution is an arcsine law (see (Feller, 1968, §III.4))

$$\frac{W_n}{n} \xrightarrow{d} W, \quad \text{where} \quad \mathbb{P}(W < x) = \frac{2}{\pi} \arcsin \sqrt{x}.$$

The corresponding probability generating function is a polynomial with only unit roots. Indeed, following the same proof as in Turán (1949), we can show that  $\mathbb{E}(z^{W_n})$  is connected to Legendre polynomials by the relation

$$\begin{aligned}
\mathbb{E}(z^{W_n}) &= [v^n] \frac{1}{\sqrt{(1-v)(1-zv)}} \\
&= z^{n/2} \text{Legendre}_n \left( \frac{z^{1/2} + z^{-1/2}}{2} \right),
\end{aligned}$$

so that the root-unitarity of the left-hand side follows from the property that Legendre polynomials have only real roots over the interval  $[-1, 1]$ . Note that the moment generating function of the arcsin law with zero mean and unit variance is given by the Bessel function

$$\begin{aligned}
\mathbb{E}(e^{(W-1/2)s/\sqrt{2}}) &= e^{-\sqrt{2}s} \left( 1 + \sum_{k \geq 1} \binom{2k}{k} \frac{(s/\sqrt{2})^k}{k!} \right) \\
&= J_0(\sqrt{2}is) = \prod_{j \geq 1} \left( 1 + \frac{2s^2}{\zeta_{0,j}^2} \right),
\end{aligned}$$

where the  $\zeta_{0,j}$ 's are the positive zeros of  $J_0(z)$ . So we have (4) with  $q = 0$  and  $q_j = 4\zeta_{0,j}^{-2}$ .



In a more general manner, from the Gegenbauer polynomials, one can also define the random variables  $W_n$  by

$$\begin{aligned}\mathbb{E}(z^{W_n}) &= \frac{1}{\binom{2\alpha+n-1}{n}} [v^n] \frac{1}{(1-v)^{-\alpha}(1-zv)^{-\alpha}} \\ &= \sum_{0 \leq j \leq n} \frac{\binom{\alpha+j-1}{j} \binom{\alpha+n-j-1}{n-j}}{\binom{2\alpha+n-1}{n}} z^j \quad (\alpha > 0),\end{aligned}$$

for which all coefficients are positive and  $\mathbb{E}(z^{W_n})$  has only unit roots. The limit law  $W_\alpha$  can be derived as in the bounded case of the Turán-Fejér polynomials

$$\begin{aligned}\mathbb{E}(e^{(W_\alpha-1/2)s}) &= \left(\frac{is}{4}\right)^{-\alpha+1/2} \Gamma(\alpha+1/2) J_{\alpha-1/2}(is/2) \\ &= \prod_{j \geq 1} \left(1 + \frac{s^2}{4\zeta_{\alpha+1/2,j}^2}\right).\end{aligned}$$

Note that the random variable  $2\sqrt{2\alpha+1}(W_\alpha-1/2)$  has variance one.

For other potential examples, see (Johnson et al., 2005, Chapter 6).

### 5.3 Uniform distribution

The literature abounds with criteria for the root-unity of polynomials. Among these, Lakatos (2002) proved that a complex polynomial  $P(z) := \sum_{0 \leq k \leq n} a_k z^k$  with  $a_k = a_{n-k}$  is root-unitary if

$$|a_n| \geq \sum_{0 \leq j \leq n} |a_n - a_j|;$$

see also Schinzel (2005). In particular, if the coefficients of  $P(z)$  are close to a constant, then all its roots lie on the unit circle. For example, let  $E_j = j![z^j]1/\cosh(z)$  denote Euler's numbers; then the polynomial

$$P_n(z) = (-1)^n \sum_{0 \leq j \leq n} \binom{2n}{2j} E_{2j} E_{2n-2j} z^j = [w^n] \frac{1}{\cos(\sqrt{w}) \cos(\sqrt{wz})},$$

is root-unitary (see Lalin and Rogers (2011)) with non-negative coefficients. See also Lalin and Smyth (2012) for more information and other root-unitary polynomials. Observe that

$$\frac{(-1)^n}{(2n)!} \binom{2n}{2j} E_{2j} E_{2n-2j} \sim \frac{4^{n+2}}{\pi^{2n+2}},$$

as  $j, n-j \rightarrow \infty$ . Thus we can show that the random variables associated with the coefficients of  $P_n(z)$  will be close to uniform, and the limit law is also uniform. Details are omitted here.

## References

Akyıldız, E. (2004). On the factorization of the Poincaré polynomial: a survey. *Serdica Math. J.*, 30(2-3):159–176.

- Andrews, G. E. (1976). *The Theory of Partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam.
- Bell, E. T. (1938). The iterated exponential integrals. *Ann. of Math.* (2), 39(3):539–557.
- Böhm, W. and Katzenbeisser, W. (2005). On the number of inversions in bimodal permutations. *J. Statist. Plann. Inference*, 135(1):55–63.
- Brenti, F. (1994). Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update. In *Jerusalem Combinatorics '93*, volume 178 of *Contemp. Math.*, pages 71–89. Amer. Math. Soc., Providence, RI.
- Canfield, E. R., Janson, S., and Zeilberger, D. (2011). The mahonian probability distribution on words is asymptotically normal. *Adv. in Appl. Math.*, 46(1-4):109–124. Supplementary material available online.
- Canfield, E. R., Janson, S., and Zeilberger, D. (2012). Corrigendum to “The Mahonian probability distribution on words is asymptotically normal” [Adv. in Appl. Math. 46 (1–4) (2011) 109–124]. *Adv. in Appl. Math.*, 49(1):77.
- Chen, W. Y. C., Wang, C. J., and Wang, L. X. W. (2008). The limiting distribution of the coefficients of the  $q$ -Catalan numbers. *Proc. Amer. Math. Soc.*, 136(11):3759–3767.
- Chern, H.-H., Hwang, H.-K., and Tsai, T.-H. (2002). An asymptotic theory for Cauchy-Euler differential equations with applications to the analysis of algorithms. *J. Algorithms*, 44:177–225.
- Comtet, L. (1974). *Advanced Combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition. The art of finite and infinite expansions.
- Cronholm, J. and Revusky, S. (1965). A sensitive rank test for comparing the effects of two treatments on a single group. *Psychometrika*, 30(4):459–467.
- d’Ocagne, M. (1887). Sur une classe de nombres remarquables. *American Journal of Mathematics*, 9(4):pp. 353–380.
- Fejér, L. (1937). Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge. *Acta. Sci. Math. Szeged*, 8:89–115.
- Feller, W. (1945). The fundamental limit theorems in probability. *Bull. Amer. Math. Soc.*, 51:800–832.
- Feller, W. (1968). *An Introduction to Probability Theory and Its Applications. Vol. I*. Third edition. John Wiley & Sons Inc., New York.
- Flajolet, P. and Sedgewick, R. (2009). *Analytic Combinatorics*. Cambridge University Press, Cambridge.
- Haigh, J. (1971). A neat way to prove asymptotic normality. *Biometrika*, 58:677–678.
- Harper, L. (1967). Stirling behavior is asymptotically normal. *Ann. Math. Stat.*, 38:410–414.

- Hayman, W. K. (1956). A generalisation of Stirling's formula. *J. Reine Angew. Math.*, 196:67–95.
- Janson, S. (1988). Normal convergence by higher semi-invariants with applications to sums of dependent random variables and random graphs. *Ann. Probab.*, 16(1):305–312.
- Johnson, N. L., Kemp, A. W., and Kotz, S. (2005). *Univariate Discrete Distributions*. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition.
- Kedlaya, K. S. (2008). Search techniques for root-unitary polynomials. In *Computational Arithmetic Geometry*, pages 71–81. Amer. Math. Soc., Providence, RI.
- Kowalenko, V. (2010). Properties and applications of the reciprocal logarithm numbers. *Acta Appl. Math.*, 109(2):413–437.
- Lakatos, P. (2002). On zeros of reciprocal polynomials. *Publ. Math. Debrecen*, 61(3-4):645–661.
- Lalin, M. and Rogers, M. (2011). Variations of the Ramanujan polynomials and remarks on  $\zeta(2j+1)/\pi^{2j+1}$ . *arXiv preprint arXiv:1106.1189*.
- Lalin, M. and Smyth, C. (2012). Unimodularity of zeros of self-inversive polynomials. *Acta Math. Hungar.*, pages 1–17.
- Louchard, G. and Prodinger, H. (2003). The number of inversions in permutations: a saddle point approach. *J. Integer Seq.*, 6(2):Article 03.2.8, 19 pp. (electronic).
- Mann, H. B. and Whitney, D. R. (1947). On a test of whether one of two random variables is stochastically larger than the other. *Ann. Math. Statistics*, 18:50–60.
- Margolius, B. H. (2001). Permutations with inversions. *J. Integer Seq.*, 4(2):Article 01.2.4, 13 pp. (electronic).
- Milovanović, G. V., Mitrinović, D. S., and Rassias, T. M. (1994). *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*. World Scientific Publishing Co. Inc., River Edge, NJ.
- Newman, C. M. (1974). Zeros of the partition function for generalized Ising systems. *Comm. Pure Appl. Math.*, 27:143–159.
- Olds, E. G. (1952). A note on the convolution of uniform distributions. *Ann. Math. Statistics*, 23:282–285.
- Park, S. (1994). The  $r$ -multipermutations. *J. Combin. Theory Ser. A*, 67(1):44–71.
- Pitman, J. (1997). Probabilistic bounds on the coefficients of polynomials with only real zeros. *J. Combin. Theory Ser. A*, 77(2):279–303.
- Policello, G. and Hettmansperger, T. (1976). Adaptive robust procedures for the one-sample location problem. *J. Amer. Statist. Assoc.*, 71(355):624–633.
- Reimer, M. (1969). Classes of semidefinite Peano-kernels. *Math. Z.*, 108:105–120.

- Rényi, A. (1972). Probabilistic methods in analysis. II. *Select. Translat. Math. Statist. Probab.*, 10:97–122.
- Sachkov, V. N. (1997). *Probabilistic Methods in Combinatorial Analysis*, volume 56. Cambridge University Press, Cambridge. Translated from the Russian, Revised by the author.
- Schinzel, A. (2005). Self-inversive polynomials with all zeros on the unit circle. *Ramanujan J.*, 9(1-2):19–23.
- Sedgewick, R. (1980). *Quicksort*. Garland Pub.
- Sheil-Small, T. (2002). *Complex Polynomials*, volume 75 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.
- Suzuki, M. (2012). On zeros of self-reciprocal polynomials. *arXiv preprint arXiv:1211.2953*.
- Takács, L. (1986). Some asymptotic formulas for lattice paths. *J. Statist. Plann. Inference*, 14(1):123–142.
- Titchmarsh, E. (1975). *The Theory of Functions*. 2nd ed. London: Oxford University Press.
- Turán, P. (1949). Remark on a theorem of Fejér. *Publ. Math. Debrecen*, 1:95–97.
- van de Wiel, M. A., Di Bucchianico, A., and van der Laan, P. (1999). Symbolic computation and exact distributions of nonparametric test statistics. *J. Roy. Statist. Soc. Ser. (The Statistician)*, 48(4):507–516.
- Wilcoxon, F. (1947). Probability tables for individual comparisons by ranking methods. *Biometrics*, 3:119–122.

## Appendix. A class of mixtures of hypergeometric distributions

Yet another class of polynomials with a similar nature to those of Turán-Fejér arises from the analysis of the partitioning stages of quicksort and defined as follows (see [Chern et al. \(2002\)](#)). Consider the random variable  $Y_n$  defined by

$$\mathbb{P}(Y_n = k) = \sum_{0 \leq j < r} p_j \frac{\binom{k}{j} \binom{n-1-k}{r-1-j}}{\binom{n}{r}},$$

where  $r \geq 1$  and  $\sum_{0 \leq j < r} p_j = 1$  is a given known distribution. Many concrete examples are discussed in [Chern et al. \(2002\)](#). Let  $P(z) := \sum_{0 \leq j < r} p_j z^j$ . Assume throughout that  $p_j = p_{r-1-j}$  for  $1 \leq j < r$ . Numerical evidence suggested that the probability generating function

$$\mathbb{E}(z^{Y_n}) = \sum_{0 \leq j < r} p_j \sum_k \frac{\binom{k}{j} \binom{n-1-k}{r-1-j}}{\binom{n}{r}} z^k$$

are root-unitary for many natural choices of  $\{p_j\}$ , but it is unclear for which class of polynomials  $P(z)$  will the polynomials be root-unitary<sup>2</sup>. The results below do not depend on the root-unitarity of  $\mathbb{E}(z^{Y_n})$ .

Assuming from now on  $p_j = p_{r-1-j}$  for  $0 \leq j < r$ , we examine the moment structure of  $Y_n$ . Note that this assumption implies that  $\sum_{0 \leq j < r} jp_j = (r-1)/2$ .

**Lemma 5.4.** *The  $m$ -th moment of  $Y_n$  is given explicitly by*

$$\mathbb{E}(Y_n^m) = \sum_{0 \leq h \leq m} \nu_{m,h} \frac{\binom{n+h}{r+h}}{\binom{n}{r}}. \quad (39)$$

Here

$$\nu_{m,h} := (-1)^{m+h} S(m+1, h+1) \sum_{0 \leq \ell \leq h} s(h+1, \ell+1) \pi_\ell,$$

where the  $s(m, h)$  denote the signless Stirling numbers of the first kind, and  $\pi_\ell := \sum_{0 \leq j < r} p_j j^\ell$ . In particular,  $\mathbb{E}(Y_n) = (n-1)/2$  and the variance satisfies

$$\mathbb{V}(Y_n) = \frac{(4\pi_2 - r^2 + 3r)n^2 + 2(6\pi_2 - 2r^2 + 3r - 1)n + 8\pi_2 - 3r^2 + 3r - 2}{4(r+1)(r+2)}.$$

Note that  $\pi_2 - 4r^2 + 3r$  is always positive because

$$\frac{r(r-3)}{4} < \pi_1^2 = \frac{(r-1)^2}{4} \leq \pi_2.$$

Thus the limit law of  $Y_n$  is never Gaussian for finite  $r$ .

*Proof.* By definition,

$$\mathbb{E}(Y_n^m) = \sum_{0 \leq j < r} p_j \sum_k k^m \frac{\binom{k}{j} \binom{n-1-k}{r-1-j}}{\binom{n}{r}}. \quad (40)$$

The, using the relation

$$k^m = \sum_{0 \leq h \leq m} (-1)^{m+h} S(m+1, h+1) (k+1) \cdots (k+m),$$

we obtain

$$\begin{aligned} & \sum_k k^m \binom{k}{j} \binom{n-1-k}{r-1-j} \\ &= \sum_{0 \leq h \leq m} (-1)^{m+h} S(m+1, h+1) \sum_k (k+1) \cdots (k+h) \binom{k}{j} \binom{n-1-k}{r-1-j} \\ &= \sum_{0 \leq h \leq m} (-1)^{m+h} S(m+1, h+1) (j+1) \cdots (j+h) \sum_k \binom{k+h}{j+h} \binom{n-1-k}{r-1-j} \\ &= \sum_{0 \leq h \leq m} (-1)^{m+h} S(m+1, h+1) (j+1) \cdots (j+h) \binom{n+h}{r+h}. \end{aligned}$$

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<sup>2</sup>The problem can be formulated by asking for which class of polynomials  $P(z) = \sum_{0 \leq j < r} p_j z^j$  will the polynomials

$$[w^n] \frac{1}{(1-w)(1-zw)} P\left(\frac{1}{(1-w)(1-zw)}\right)$$

have only unit roots?

Now, by substituting the expression

$$(j+1) \cdots (j+h) = \sum_{0 \leq \ell \leq h} s(h+1, \ell+1) j^\ell,$$

We then obtain (39). ■

**Theorem 5.5.** *The sequence of random variables  $\{Y_n/n\}$  converges in distribution to a limit law  $Y$  whose moment generating function satisfies*

$$\mathbb{E}(e^{Ys}) = r \sum_{0 \leq j < r} p_j \binom{r-1}{j} \int_0^1 e^{xs} x^{r-1-j} (1-x)^j dx. \quad (41)$$

*Proof.* Indeed, by (39),

$$\frac{\mathbb{E}(Y_n^m)}{n^m} \sim \frac{r!}{(r+m)!} \sum_{0 \leq j < r} p_j (j+1) \cdots (j+m),$$

so that

$$\frac{\mathbb{E}(Y_n^m)}{n^m} \xrightarrow{d} Y,$$

where the moment generating function of  $X$  is given by

$$\begin{aligned} \mathbb{E}(e^{Ys}) &= \sum_{m \geq 0} \frac{s^m}{m!} \sum_{0 \leq j < r} p_j \frac{\Gamma(j+m+1)\Gamma(r+1)}{\Gamma(r+m+1)\Gamma(j+1)} \\ &= \sum_{m \geq 0} \frac{s^m}{m!} \sum_{0 \leq j < r} p_j \frac{\Gamma(r-j+m)\Gamma(r+1)}{\Gamma(r+m+1)\Gamma(r-j)} \\ &= r \sum_{m \geq 0} \frac{s^m}{m!} \sum_{0 \leq j < r} p_j \binom{r-1}{j} \int_0^1 (1-x)^j x^{r-1-j+m} dx, \end{aligned}$$

which proves (41). The justification of the unique characterization of this limit law is straightforward. ■

Examples.

- The median of  $(2k+1)$  elements: This corresponds to the case when  $r = 2k+1$  and  $p_k = 1$ . We then obtain

$$\mathbb{E}(e^{Ys}) = \frac{(2k+1)!}{k!k!} \int_0^1 e^{xs} x^k (1-x)^k dx,$$

a Beta distribution (with a Bessel-type infinite-product representation); see (33) and (34).

- Uniform distributon: In this case,  $p_j = 1/r$ ,  $1 \leq j < r$ . This is algorithmically uninteresting, but has the limit moment generating function  $(e^s - 1)/s$ .
- The ninther (the median of three medians, each being the median of three elements): This is the case when

$$\{p_j\}_{j=0,\dots,8} = \left\{0, 0, 0, \frac{3}{14}, \frac{4}{7}, \frac{3}{14}, 0, 0, 0\right\}.$$

We have a mixture of Beta distributions for the limit law.

Many sophisticated cases can be found in Chern et al. (2002).